Lecture 19

Module III - Micro-mechanics of Lamina

Micro-mechanics of Lamina

Micromechanics deals with the study of composite material behaviour in terms of the interaction of its constituents. From the procedures of micromechanics lamina properties can be predicted. There are two basic approaches of the micromechanics of composite materials, namely (i) Mechanics of materials and (ii) Elasticity

Here, the mechanics of materials approach will be followed.

Volume Fractions:

Consider a composite material that consists of fibers and matrix material. The volume of the composite material is equal to the sum of the volume of the fibers and the volume of the matrix. Therefore,

$$v_c = v_f + v_m \tag{3.142}$$

where, ν_c - volume of composite material

ν_f - volume of fiber

ν_m - volume of matrix

Let, the fiber volume fraction V_f and the matrix volume fraction V_m be defined as

$$V_f = \frac{\nu_f}{\nu_c} \quad \text{and} \tag{3.143}$$

$$V_m = \frac{\nu_m}{\nu_c} \tag{3.144}$$

such that the sum of volume fractions is

$$V_f + V_m = 1 \tag{3.145}$$

Weight Fractions:

Assuming that the composite material consists of fibers and matrix material, the weight of the composite material is equal to the sum of the weight of the fibers and the weight of the matrix. Therefore,

$$w_c = w_f + w_m \tag{3.146}$$

where, w_c - weight of composite material

w_f - weight of fiber

w_m - weight of matrix

The weight fractions (mass fractions) of the fiber and the matrix are defined as

$$W_f = \frac{w_f}{w_c} \quad \text{and} \tag{3.147}$$

$$W_m = \frac{w_m}{w_c} \tag{3.148}$$

such that the sum of weight fractions is

$$W_f + W_m = 1 \tag{3.149}$$

Density:

The density of composite material can be defined as the ratio of weight of the composite material to the volume of the composite material and is expressed as

$$\rho_c = \frac{w_c}{v_c} \tag{3.150}$$

but, $v_c = v_f + v_m$, and $v = \frac{w}{\rho}$, therefore the above equation can be rewritten as

$$\frac{w_c}{\rho_c} = \frac{w_f}{\rho_f} + \frac{w_m}{\rho_m} \tag{3.151}$$

$$\frac{w_c}{\rho_c} = \frac{w_f}{\rho_f} + \frac{w_m}{\rho_m}$$

$$\frac{1}{\rho_c} = \frac{1}{\rho_f}\left(\frac{w_f}{w_c}\right) + \frac{1}{\rho_m}\left(\frac{w_m}{w_c}\right) \tag{3.152}$$

By writing in terms of weight fractions,

$$\frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m} \quad (3.153)$$

$$\frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}$$

The density of the composite material in terms of weight fractions can be written as

$$\rho_c = \frac{1}{\left(\frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}\right)} \quad (3.154)$$

$$\rho_c = \frac{1}{\sum_{i=1}^{n}\left(\frac{W_i}{\rho_i}\right)} \quad (3.155)$$

Moreover, the equation $w_c = w_f + w_m$, in general, can be rewritten as

$$\rho_c v_c = \rho_f v_f + \rho_m v_m$$

$$\rho_c = \rho_f\left(\frac{v_f}{v_c}\right) + \rho_m\left(\frac{v_f}{v_c}\right) \quad (3.156)$$

writing in terms of volume fractions, the density of the composite material is written as

$$\rho_c = \rho_f V_f + \rho_m V_m \quad (3.157)$$

In general,

$$\rho_c = \sum_{i=1}^{n} \rho_i V_i \quad (3.158)$$

Void Content:

During the incorporation of fibers into the matrix or during the manufacturing of laminates, air or other volatiles may be trapped in the material. The trapped air or volatiles exist in the laminate as micro voids, which may significantly affect some of its mechanical properties. A high void content (over 5% by volume) usually leads to lower fatigue resistance, greater susceptibility to water diffusion, and increased variation (scatter) in mechanical properties. The void content in a composite laminate can be estimated by comparing the theoretical density with its actual density.

$$v_{void} = \left(\frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}} \right) * 100 \quad (3.159)$$

where, ρ_{ct} - theoretical density of the composite material

ρ_{ce} - experimental density of the composite material

Determination of Longitudinal Modulus:

Consider a unidirectional composite specimen as shown in Fig.3.22.

The following assumptions are made to get the basic properties:

(i) Fibers are uniform in properties and diameter.

(ii) Fibers are continuous and parallel throughout the composites.

(iii) There is a perfect bonding between the fibers and the matrix.

(iv) Strains experienced by the fiber, matrix and composites are equal. i.e.

$$\varepsilon_c = \varepsilon_f = \varepsilon_m$$

where, $\varepsilon_c, \varepsilon_f$ *and* ε_m are the longitudinal strains in fibers, matrix, and composite respectively. This condition is called iso-strain condition.

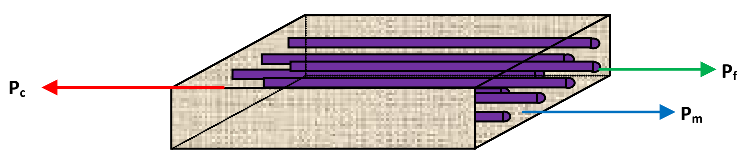


Figure 3.22 Model for longitudinal behaviour of composite material

Let, the composite be applied by a load P_c which is shared between the fibers and the matrix so that

$$P_c = P_f + P_m \quad (3.160)$$

The corresponding stress relation is

$$\sigma_c A_c = \sigma_f A_f + \sigma_m A_m \quad (3.161)$$

Further its behaviour is assumed to be linearly elastic, and hence the modulus and stress are related. Thus,

$$E_c\,\varepsilon_c A_c = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m \tag{3.162}$$

$$E_c = E_f\left(\frac{\varepsilon_f A_f}{\varepsilon_c A_c}\right) + E_m\left(\frac{\varepsilon_m A_m}{\varepsilon_c A_c}\right) \tag{3.163}$$

But, for parallel fibers the area fraction is same as the volume fraction. Thus,

$$E_c = E_f V_f + E_m V_m \tag{3.164}$$

The relationship of this form is known as ***Rule or Law of Mixtures***

$$\mathbf{E_c = E_f V_f + E_m (1 - V_f)} \tag{3.165}$$

In general

$$\mathbf{E_c = \sum_{i=1}^{n} E_i V_i} \tag{3.166}$$

Longitudinal strength:

The load is shared by the fibers and the matrix.

$$P_c = P_f + P_m \tag{3.167}$$

Thus,

$$\sigma_c A_c = \sigma_f A_f + \sigma_m A_m \tag{3.168}$$

$$\sigma_c = \sigma_f\left(\frac{A_f}{A_c}\right) + \sigma_m\left(\frac{A_m}{A_c}\right) \tag{3.169}$$

$$\sigma_c = \boldsymbol{\sigma_f V_f + \sigma_m V_m} \tag{3.170}$$

$$\sigma_c = \sigma_f V_f + \sigma_m (1 - V_f) \tag{3.171}$$

Load carrying capacity of fibers:

The strains experienced by the composites, fibers and the matrix are equal. Thus,

$$\varepsilon_c = \varepsilon_f = \varepsilon_m \tag{3.172}$$

$$\frac{\sigma_c}{E_c} = \frac{\sigma_f}{E_f} = \frac{\sigma_m}{E_m} \tag{3.173}$$

$$\frac{\sigma_f}{\sigma_m} = \frac{E_f}{E_m} \quad \text{and} \quad \frac{\sigma_f}{\sigma_c} = \frac{E_f}{E_c} \tag{3.174}$$

$$\frac{P_f}{P_m} = \frac{\sigma_f A_f}{\sigma_m A_m} = \frac{\frac{([E_f \varepsilon_f) A]_f}{Ac}}{\frac{(E_m \varepsilon_m) A_m}{Ac}} = \frac{E_f V_f}{E_m V_m} \quad (3.175)$$

$$\frac{P_f}{P_c} = \frac{\sigma_f A_f}{\sigma_f A_f + \sigma_m A_m} \quad (3.176)$$

$$\frac{P_f}{P_c} = \frac{\sigma_f A_f \frac{}{Ac}}{\frac{[(\sigma]_f A_f + \sigma_m A_m)}{Ac}} = \frac{\sigma_f V_f}{\sigma_f V_f + \sigma_m V_m} \quad (3.177)$$

$$\frac{P_f}{P_c} = \frac{E_f \varepsilon_f V_f}{E_f \varepsilon_f V_f + E_m \varepsilon_m V_m} \quad (3.178)$$

$$\frac{\mathbf{P_f}}{\mathbf{P_c}} = \frac{\mathbf{E_f V_f}}{\mathbf{E_f V_f + E_m V_m}} \quad (3.179)$$

or,

$$\frac{\mathbf{P_f}}{\mathbf{P_c}} = \frac{\frac{\mathbf{E_f}}{\mathbf{E_m}}}{\frac{\mathbf{E_f}}{\mathbf{E_m}} + \frac{\mathbf{V_m}}{\mathbf{V_f}}} \quad (3.180)$$

Thus, the load sharing of the fibers depend on the modulus values and the volume fractions of fiber and matrix.

Determination of Transverse Modulus:

Consider unidirectional composites. The following assumptions are made

(i) Fibers are uniform in properties and diameter.

(ii) Fibers are continuous and parallel throughout the composite.

(iii)There is a perfect bonding between the fibers and the matrix.

(iv)The fibers and the matrix are made up of layers and each layer will carry the same load. Therefore the fiber and matrix layers will experience equal stress. i.e.

$$\sigma_c = \sigma_f = \sigma_m \quad (3.181)$$

where, $\sigma_c, \sigma_f,$ **and** σ_m are the stresses in the composites, fibers and matrix respectively, in the loading direction (transverse direction).

(v) the thickness of the composite material is equal to the sum of the thickness of fibers and matrix. i.e.

$$t_c = t_f + t_m \quad (3.182)$$

where, t_c, t_f, and t_m are the thicknesses of composites, fibers, and matrix respectively

Let, the load be applied in the transverse direction, i.e. the direction perpendicular to the parallel fibers.

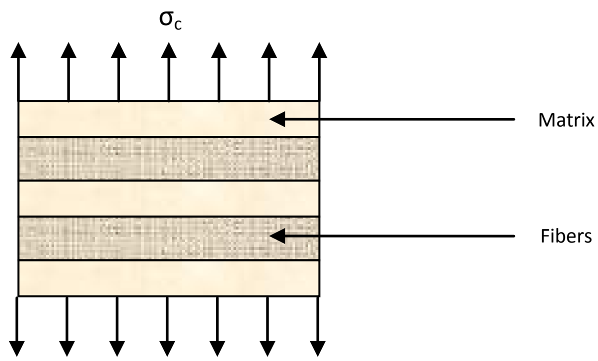


Figure 3.23 Model for transverse behaviour of composite material

Under the applied transverse load, the elongation of composite material δ_c in the direction of the load is the sum of the fiber elongation and the matrix elongation. i.e.

$$\delta_c = \delta_f + \delta_m \tag{3.183}$$

where, δ_c, δ_f, *and* δ_m are the elongations of composite, fibers, and matrix respectively

but , strain, $\varepsilon = \delta / t$, (3.184)

and it gives $\delta = \varepsilon * t$ (3.185)

therefore, the above equation can be rewritten as

$$\varepsilon_c t_c = \varepsilon_f t_f + \varepsilon_m t_m \tag{3.186}$$

$$\varepsilon_c = \varepsilon_f \left(\frac{t_f}{t_c}\right) + \varepsilon_m \left(\frac{t_m}{c}\right) \tag{3.187}$$

$$\varepsilon_c = \varepsilon_f V_f + \varepsilon_m V_m$$

$$\frac{\sigma_c}{E_c} = \frac{\sigma_f}{E_f} V_f + \frac{\sigma_m}{E_m} V_m \tag{3.188}$$

but, $\sigma_c = \sigma_f = \sigma_m$ (3.189)

therefore,

$$\frac{1}{E_{cT}} = \frac{V_f}{E_f} + \frac{V_m}{E_m} \tag{3.190}$$

$$E_{cT} = \frac{E_f * E_m}{E_f V_m + E_m V_f} \tag{3.191}$$

or

$$E_{cT} = \frac{1}{\Sigma_{i=1}^{n} \frac{V_i}{E_i}} \tag{3.192}$$

Exercise Problems:

1) Find the weight fraction and volume fraction of fibers in the glass/epoxy composites . The following data is obtained from the burnout test.

weight of the empty crucible = 46.5401 gm

weight of crucible and composite piece = 49.1201 gm

weight of crucible and glass fiber = 48.3420 gm.

The density of glass fiber is 2600 kg/m ^{3}

and 1300 kg/m ^{3}

2.) Calculate the ratio of fiber stress to matrix stress and matrix stress to composite stress for V_f = 15%, 30 %, 45 % and 70 %. Take E_f =250 G Pa and E_m = 15 G Pa.

References:

1) "Analysis and Performance of Fiber composites", BD Agarwal, L J Broughtman and K Chandrashekhara, John Wiley and sons.

2) " Principles of Composite Material Mechanics, Ronald R Gibson, CRC Press.

Lecture 20 **Short fibers**

Theories of stress transfer in short fibers:

Tensile load applied to a discontinuous fiber lamina is transferred to the fibers by a shearing mechanism between fibers and matrix. Since, the matrix has low modulus, the longitudinal strain in the matrix is higher than that in the adjacent fibers. If a perfect bond is assumed between the two constituents, the difference in longitudinal strains creates a shear stress distribution across the fiber–matrix interface. Ignoring the stress transfer at the fiber end cross sections and the interaction between the neighboring fibers, we can calculate the normal stress distribution in a discontinuous fiber by a simple force equilibrium analysis.

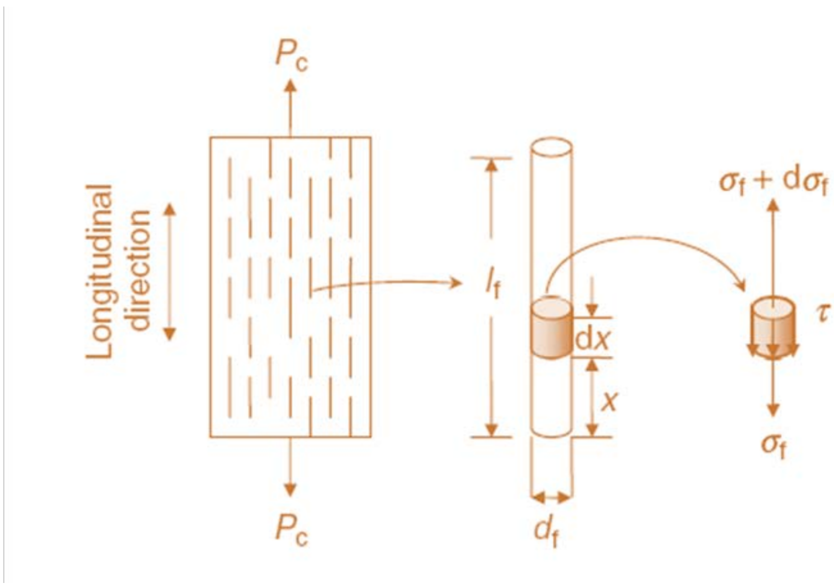


Figure 3.24 stress distribution on short fiber

Consider an infinitesimal length dx at a distance x from one of the fiber ends. The force equilibrium equation for this length is

$$(\pi r^2)\sigma_f + (2\pi r\ dz)\tau = (\pi r^2)(\sigma_f + d\sigma_f) \qquad (3.193)$$

which on simplification gives

$$\frac{d\sigma_f}{dz} = \frac{2\tau}{r} \qquad (3.194)$$

where,

σ_f is the fiber stress in the axial direction

τ is the shear stress on the cylindrical fiber-matrix interface

r is the fiber radius

$$\sigma_f = \sigma_{fo} + \frac{2}{r}\int_0^z \tau \, dz \tag{3.195}$$

where, σ_{fo} is the stress on the fiber end. In many analyses σ_{fo} is neglected because of yielding of the matrix adjacent to the fiber end or separation of the fiber end from the matrix as a result of large stress concentrations. Therefore,

$$\sigma_f = \frac{2}{r}\int_0^z \tau \, dz = \frac{2\tau_z}{r} \tag{3.196}$$

The maximum fiber stress occurs at the midfiber length, *i.e.*, at $z = \frac{l}{2}$

$$(\sigma_f)_{max} = \frac{\tau l}{r} \tag{3.197}$$

Based on the assumption that the strains in fibers, matrix and composite are equal, the maximum fiber stress $(\sigma_f)_{max}$ is limited as given below.

$$\varepsilon_c = \varepsilon_f \tag{3.198}$$

$$\frac{(\sigma_f)_{max}}{E_f} = \frac{\sigma_c}{E_c} \tag{3.199}$$

$$(\sigma_f)_{max} = \frac{E_f}{E_c}\sigma_c \tag{3.200}$$

The minimum fiber length may be defined as a load-transfer length in which the maximum fiber stress, $\sigma_{f,max}$ can be achieved on the application of the external load, σ_c.

The minimum fiber length, $l_t = \frac{\sigma_{f,\ max}}{\tau}\frac{d}{2} = \frac{(E_f/(E_c)\sigma c^d)}{2\tau}$ (3.201)

where, 'd' is the diameter of the fiber

The critical fiber length may be defined as the minimum fiber length in which the maximum allowable fiber stress or the fiber ultimate strength, σ_u can be achieved.

The critical fiber length, $l_c = \frac{\sigma_f u}{\tau}\frac{d}{2}$ (3.202)

Thus the minimum fiber length, l_t, is based on the applied stress, σ_c, whereas the critical fiber length, l_c, is independent of applied stress, σ_c.

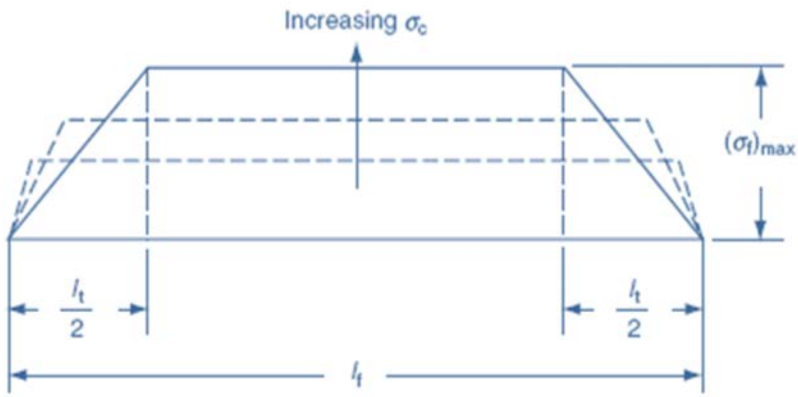


Figure 3.25a stress on short fiber

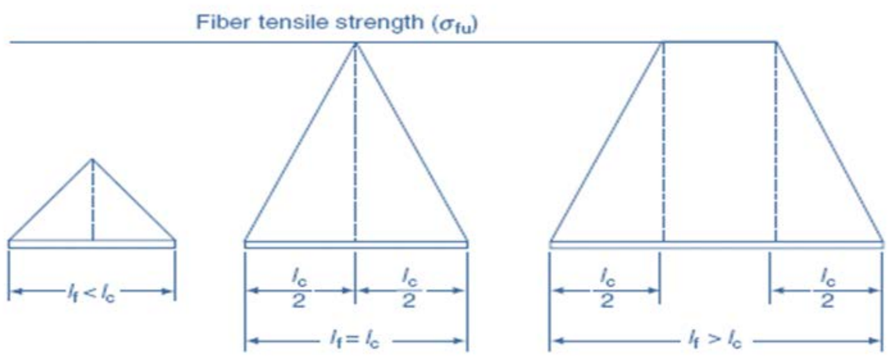


Figure 3.25b Variations of fiber stress for different fiber lengths

From the figure 3.25, the following points are deduced.

(i) For $l_f < l_c$, the maximum fiber stress may never reach the ultimate fiber strength. In this case, either the fiber–matrix interfacial bond or the matrix may fail before fibers achieve their ultimate strength.

(ii) For $l_f > l_c$, the maximum fiber stress may reach the ultimate fiber strength over much of its length. However, over a distance equal to l_c /2 from each end, the fiber remains less effective.

(iii) For effective fiber reinforcement, that is, for using the fiber to its ultimate strength, one must select $l_f >> l_c$.

(iv) For a given fiber diameter and strength, l_c can be controlled by increasing or decreasing τ. For example, a matrix-compatible coupling agent may increase τ, which in turn decreases l_c. If l_c can be reduced relative to l_f through proper fiber surface treatments, effective reinforcement can be achieved without changing the fiber length.

Prediction of Modulus of short fibers:

The Halpin-Tsai equations provide the way to calculate various moduli of aligned short-fiber composites.

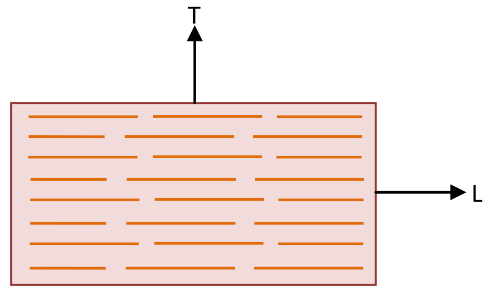


Figure 3.26 Model of an aligned short-fiber compsite

$$E_L = \frac{1+2\left(\frac{l_f}{d_f}\right)\eta_L V_f}{1-\eta_L V_f} E_m \quad (3.203)$$

$$E_T = \frac{1+2\eta_T V_f}{1-\eta_T V_f} E_m \quad (3.204)$$

$$G_{LT} = \frac{1+\eta_G V_f}{1-\eta_G V_f} G_m \quad (3.205)$$

$$\nu_{LT} = V_f \nu_f + V_m \nu_m \quad (3.206)$$

where,

$$\eta_L = \frac{E_f / E_m - 1}{\frac{E_f}{E_m} + 2\left(\frac{l_f}{d_f}\right)} \quad (3.207)$$

$$\eta_T = \frac{E_f / E_m - 1}{E_f / E_m + 2} \quad (3.208)$$

$$\eta_G = \frac{G_f / G_m - 1}{G_f / G_m + 1} \quad (3.209)$$

The short fiber composite with random orientation produces the composite with isotropic behaviour in a plane. To predict the elastic moduli of such randomly oriented composites, the empirical formulae given below are used.

$$E_{random} = \frac{3}{8}E_L + \frac{5}{8}E_T \qquad (3.210)$$

$$G_{random} = \frac{1}{8}E_L + \frac{1}{4}E_T \qquad (3.211)$$

$$\nu_{random} = \frac{E_{random}}{G_{random}} - 1 \qquad (3.212)$$

Lecture 21 and 22

Problems (Module III):

Problem 3.1: Calculate the fraction of load carried by the fibers of glass-epoxy composites with 30% fibers by volume. Elastic moduli of glass fibers and epoxy resin are 70 and 3.5 GPa respectively.

Solution:

The formula for the load shared by fibers is given by,

$$\frac{P_f}{P_c} = \frac{E_f / E_m}{\left(E_f / E_m + \left(\frac{V_m}{V_f}\right)\right)} \qquad (3.213)$$

V_f (given) = 0.30

V_m = 1 - V_f = 1 - 0.30

= 0.70

E_f (given) = 70 GPa

E_m(given) = 3.5 GPa

$$\frac{P_f}{P_c} = \frac{70/3.5}{\left(70/3.5 + (0.7/0.3)\right)} \qquad (3.214)$$

= 0.90

Therefore, the fiber carries 90 % of the total load applied to the composite. If the volume fraction of fiber is increased, then the composite will carry more load. But, there is a limit for the maximum volume fraction of fiber, practically around 70%. If the volume fraction of fiber is more, then the entire fibers could not be wetted properly due to less amount of matrix.

Problem 3.2 : Calculate the elastic constants for the composite that consists of randomly distributed short glass fibers 60% by weight. The diameter and the length of the fiber used are 2.5 mm and 25 mm respectively. The Epoxy resin is used as matrix. Assume the necessary data if not given.

Solution:

As the fibers are short and distributed randomly, the Halpin-Tsai equations will be used to determine the young's modulus, shear modulus and Poisson's ratio.

Data given

E_f = 70 GPa (assumed)

E_m = 3.5 GPa (assumed)

ρ_f = 2.5 g/cm^3 (assumed)

ρ_m = 1.2 g/cm^3 (assumed)

l_f = 25 mm (given)

d_f = 2.5 mm (given)

w_f = 0.60 (given)

The equations used for determining young's modulus for a unidirectional lamina is given by

$$E_{11} = \frac{1+2\left(l_f / d_f\right)\eta_L V_f}{1-\eta_L V_f} E_m \qquad (3.215)$$

$$E_{22} = \frac{1+2\eta_T V_f}{1-\eta_T V_f} E_m \qquad (3.216)$$

Therefore, it is necessary to calculate the volume fraction of fibers, V_f , the coefficients η_L and η_T.

$$V_f = \frac{\frac{W_f}{\rho_f}}{\left(\frac{W_f}{\rho_f}\right)+\left(\frac{W_m}{\rho_m}\right)} \qquad (3.217)$$

$$W_m = 1 - W_f = 1 - 0.60 \qquad (3.218)$$

$$= 0.40$$

$$V_f = \frac{\frac{0.6}{2.5}}{\left(\frac{0.6}{2.5}\right)+\left(\frac{0.4}{1.2}\right)} \qquad (3.219)$$

$$= 0.42$$

$$\eta_L = \frac{\left(E_f / E_m\right)-1}{\left(E_f / E_m\right)+2\left(l_f / d_f\right)} \qquad (3.220)$$

$$= \frac{\left(70/3.5\right)-1}{\left(70/3.5\right)+2\left(25/2.5\right)}$$

$$= 0.475$$

$$\eta_T = \frac{\left(E_f / E_m\right)-1}{\left(E_f / E_m\right)+2} \qquad (3.221)$$

$$= \frac{\left(70/3.5\right)-1}{\left(70/3.5\right)+2}$$

$$= 0.864$$

Therefore,

$$E_{11} = \frac{1+2\left(25/2.5\right)*0.475*0.42}{1-0.475*0.42}*3.5 \qquad (3.222)$$

$$= 21.82 \text{ GPa}$$

$$E_{22} = \frac{1+2*0.864*0.42}{1-0.864*0.42}*3.5 \qquad (3.223)$$

$$= 9.48 \text{ GPa}$$

The equations of elastic constants for randomly oriented short fibers are given by:

$$E_{random} = \frac{3}{8}E_{11} + \frac{5}{8}E_{22} \quad (3.224)$$

$$G_{random} = \frac{1}{8}E_{11} + \frac{1}{4}E_{22} \quad (3.225)$$

$$\nu_{random} = \frac{E_{random}}{2G_{random}} - 1 \quad (3.226)$$

Therefore,

$$E_{random} = \frac{3}{8}*21.82 + \frac{5}{8}*9.48 \quad (3.227)$$

= 14.11 GPa

$$G_{random} = \frac{1}{8}*21.82 + \frac{1}{4}*9.48 \quad (3.228)$$

= 5.10 GPa

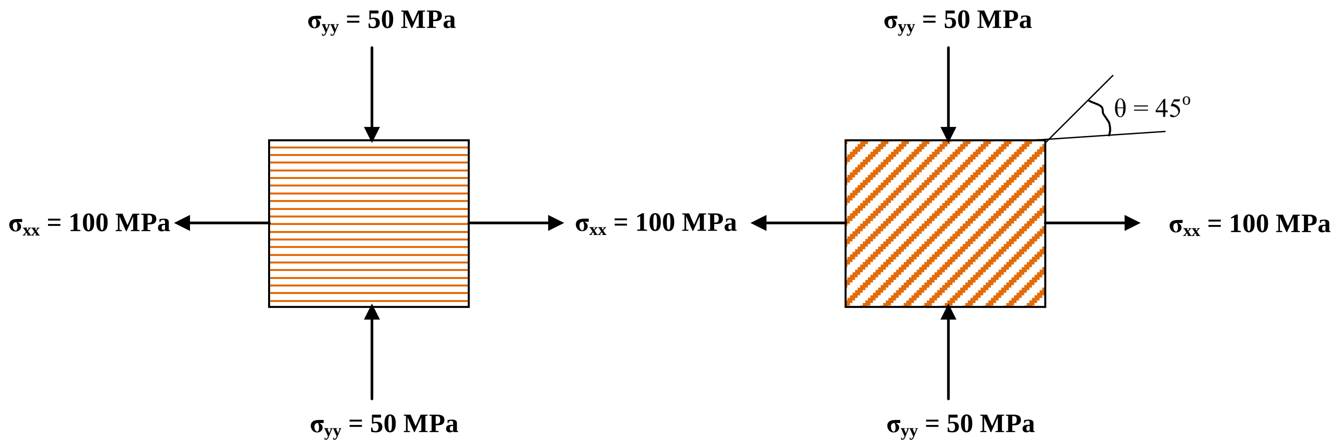
$$\nu_{random} = \frac{14.11}{2*5.10} - 1 \quad (3.229)$$

= 0.383

Problem 3.3 : Calculate the strains in the xy directions for the composite subjected to the loading as shown in the figure. The composite is made of boron-epoxy. Take the data given for 0^o unidirectional E-glass-epoxy as:

E_{11} = 200 GPa; E_{22} = 20 GPa;

G_{12} = 6.5 GPa; ν_{12} = 0.2;



case (a) case (b)

Solution:

case (a): It is a 0^o unidirectional composite. Therefore, it is a specially orthotropic lamina.

The strains experienced by the composite are given by,

$$\varepsilon_{XX} = \varepsilon_{11} = \frac{\sigma_{XX}}{E_{11}} - \nu_{21}\frac{\sigma_{yy}}{E_{22}} \quad (3.230)$$

$$\varepsilon_{yy} = \varepsilon_{22} = \frac{\sigma_{yy}}{E_{22}} - \nu_{12}\frac{\sigma_{XX}}{E_{11}} \quad (3.231)$$

$$\gamma_{Xy} = \gamma_{yx} = \gamma_{12} = \gamma_{21} = \frac{\tau_{Xy}}{G_{12}} \quad (3.232)$$

Data given:

$\sigma_{xx} = 100$ MPa (3.233)

$\sigma_{yy} = -50$ MPa (Compression) (3.234)

$\tau_{xy} = 0$

ν_{21} may be calculated from,

$$\nu_{21} = \nu_{12}\frac{E_{22}}{E_{11}} \quad (3.235)$$

$$\nu_{21} = 0.2 * \frac{20}{200}$$

$$= 0.02$$

Therefore, $\varepsilon_{Xx} = \frac{100\times10^{-3}}{200} - 0.02 * \frac{-50\times10^{-3}}{20}$ (3.236)

$= 0.55 \times 10^{-3}$

$$\varepsilon_{yy} = \frac{-50\times10^{-3}}{20} - 0.2 * \frac{100\times10^{-3}}{200} \quad (3.237)$$

$= -2.6 \times 10^{-3}$

$$\gamma_{Xy} = \frac{0}{G_{12}} \quad (3.238)$$

$= 0$

case (b): It is a 45° unidirectional composite loaded in the x and y direction, not along the principal material directions. Therefore, it is a generally orthotropic case.

The elastic constants are calculated first using,

$$\frac{1}{E_x} = \frac{\cos^4\theta}{E_L} + \frac{\sin^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta \quad (3.239)$$

$$\frac{1}{E_y} = \frac{\sin^4\theta}{E_L} + \frac{\cos^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta \quad (3.240)$$

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}}\right)\sin^2 2\theta \quad (3.241)$$

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{TL}}{E_T} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}}\right)\sin^2 2\theta \quad (3.242)$$

Therefore, $E_x = 10.34$ GPa

$E_y = 10.34$ GPa (3.243)

$\nu_{xy} = \nu_{yx} = 0.26$

The coefficients of mutual influence are calculated using,

$$m_x = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \cos^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right] \quad (3.244)$$

$$m_y = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \sin^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right] \quad (3.245)$$

$m_x = m_y = 4.5$

The strains are calculated using the equations:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_{yx}\frac{\sigma_y}{E_y} - m_x\frac{\tau_{xy}}{E_L} \quad (3.246)$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \nu_{xy}\frac{\sigma_x}{E_x} - m_y\frac{\tau_{xy}}{E_L} \quad (3.247)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} - m_x\frac{\sigma_x}{E_L} - m_y\frac{\sigma_y}{E_L} \quad (3.248)$$

$$\varepsilon_x = \frac{100\times10^{-3}}{10.34} - 0.02 * \frac{-50\times10^{-3}}{10.34} - m_x\frac{0}{E_L} \quad (3.249)$$

$= 9.768 \times 10^{-3}$

$$\varepsilon_y = \frac{-50\times10^{-3}}{10.34} - 0.2 * \frac{100\times10^{-3}}{10.34} - m_y\frac{0}{E_L} \quad (3.250)$$

$= -6.770 \times 10^{-3}$

$$\gamma_{xy} = \frac{0}{G_{xy}} - 4.5 * \frac{100\times10^{-3}}{10.34} - 4.5 * \frac{-50\times10^{-3}}{10.34} \quad (3.251)$$

$= -43.30 \times 10^{-3}$

Problem 3.4: Determine the stiffness matrix for an angle-ply graphite-epoxy lamina containing 50% volume of fibers. Take the following engineering constants for the composite. Consider the fiber orientation angles of both 0^o and 45^o.

$E_f = 230$ GPa $\quad E_m = 3.5$ GPa $\quad \nu_f = 0.2 \quad \nu_m = 0.3$

Solution:

As the engineering constants of the composite are not directly given, they are to be determined using the rule of mixtures.

$V_f = 0.50$ (given)

$$E_c = E_{11} = E_f * V_f + E_m * V_m \quad (3.252)$$

$= 230*0.5 + 3.5*(1 - 0.5)$

$= 116.75$ GPa

$$E_{22} = \frac{E_f * E_m}{E_f * V_m + E_m * V_f} \qquad (3.253)$$

$$= \frac{230 * 3.5}{230 * (1 - 0.5) + 3.5 * 0.5}$$

$= 6.90$ GPa

$$\nu_{12} = \nu_f * V_f + \nu_m * V_m \qquad (3.254)$$

$= 0.2 * 0.5 + 0.3 * (1- 0.5)$

$= 0.25$

$$\nu_{21} = \frac{E_{22}}{E_{11}} \nu_{12} \qquad (3.255)$$

$$= \frac{6.90}{116.75} * 0.25$$

$= 0.015$

As the values of G_f and G_m are not given, in order to calculate G_{12}, G_f and G_m are determined based isotropic relationship as follows:

$$G_f = \frac{E_f}{2(1+\nu_f)} \qquad (3.256)$$

$$= \frac{230}{2(1 + 0.2)}$$

$= 95.83$ GPa

$$G_m = \frac{E_m}{2(1+\nu_m)} \qquad (3.257)$$

$$G_m = \frac{3.5}{2(1+0.3)}$$

$= 1.35$ GPa

Therefore, $$G_{12} = \frac{G_f * G_m}{G_f * V_m + G_m * V_f} \quad (3.258)$$

$$= \frac{95.83 * 1.35}{95.83 * (1 - 0.5) + 1.35 * 0.5}$$

$= 2.66$ GPa

Case (a) 0^o lamina

As the fiber orientation angle is 0^o, it is a specially orthotropic case. Elements of the stiffness matrix are obtained from,

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \quad (3.259)$$

$$= \frac{116.75}{1 - 0.25 * 0.015}$$

$= 117.19$ GPa

$$Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \quad (3.260)$$

$$= \frac{6.90}{1 - 0.25 * 0.015}$$

$= 6.93$ GPa

$$Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21} E_{11}}{1 - \nu_{12}\nu_{21}} \quad (3.261)$$

$$= \frac{0.25 * 6.90}{1 - 0.25 * 0.015}$$

$= 1.73$ GPa

$$Q_{66} = G_{LT} \quad (3.262)$$

$= 2.66$ GPa

The stiffness matrix for 0^o lamina is given by

$$[Q] = \begin{bmatrix} 117.19 & 1.73 & 0 \\ 1.73 & 6.93 & 0 \\ 0 & 0 & 2.66 \end{bmatrix} \text{GPa}$$

The compliance matrix can be obtained by inversing the stiffness matrix,

$$[S] = [Q]^{-1} = \begin{pmatrix} 8.5 & -2.125 & 0 \\ -2.125 & 144.83 & 0 \\ 0 & 0 & 375.94 \end{pmatrix} \times 10^{-3} \text{ GPa -1}$$

Case (b) 45^o lamina

As the fiber orientation angle is 45^o, the lamina will experience the generally orthotropic behaviour. The elements of the transformed reduced stiffness matrix are obtained as,

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^2\theta - (Q_{22} - Q_{12} - 2Q_{66})\sin^2\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^2\theta\cos\theta - (Q_{22} - Q_{12} - 2Q_{66})\sin\theta\cos^2\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned} \quad (3.263)$$

Therefore,

$$\bar{Q}_{11} = 117.19 * \cos^4 45^\circ + 6.93\sin^4 45^\circ + 2(1.73 + 2*2.66)\sin^2 45^\circ \cos^2 45^\circ$$

$$= 34.56 \text{ GPa}$$

$$\bar{Q}_{22} = 117.19*\sin^4 45^\circ + 6.93\cos^4 45^\circ + 2(1.73 + 2*2.66)\sin^2 45^\circ \cos^2 45^\circ$$

$$= 34.56 \text{ GPa}$$

$$\overline{Q_{12}} = (117.19 + 6.93 - 4*2.66)\sin^2 45^\circ \cos^2 45^\circ + 1.73(\sin^4 45^\circ + \cos^4 45^\circ)$$

$$= 29.24 \text{ GPa}$$

$$\overline{Q}_{16} = (117.19 - 1.73 - 2*2.66)\sin 45^\circ \cos^2 45^\circ$$

$$-(6.93 - 1.73 - 2*2.66)\sin^3 45^\circ \cos 45^\circ$$

= 27.57 GPa

$$\overline{Q_{26}} = (117.19 - 1.73 - 2*2.66)\sin^3 45^\circ \cos 45^\circ$$
$$-(6.93 - 1.73 - 2*2.66)\sin 45^\circ \cos^3 45^\circ$$

= 27.57 GPa

$$\overline{Q}_{66} = (117.19 + 6.93 - 2*1.73 - 2*2.66)\sin^2 45^\circ \cos^2 45^\circ$$

$$+2.66\left(\sin^4 45^\circ + \cos^4 45^\circ\right) \quad (3.264)$$

= 31.50 GPa

The transformed reduced stiffness matrix for 45° lamina is given by

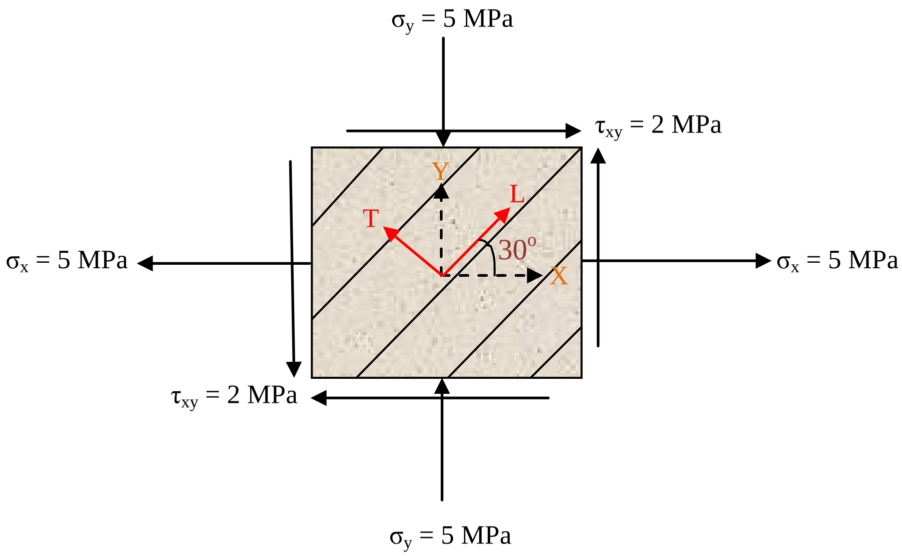
$$[\bar{Q}] = \begin{bmatrix} 34.56 & 29.24 & 27.57 \\ 29.24 & 34.56 & 27.57 \\ 27.57 & 27.57 & 31.50 \end{bmatrix} \text{GPa}$$

As expected the stiffness matrix for 45° lamina contains all non-zero elements.

Problem 3.5: Find the stresses and strains in both the principal material directions and the reference directions (xy) for the lamina shown in the figure. The engineering constants for the lamina may be taken as:

E_L = 13.8 GPa E_T = 3.35 GPa G_{LT} = 4.12 GPa

ν_{LT} = 0.36 ν_{TL} = 0.087



Solution:

Data given:

$\sigma_x = 5$ MPa

$\sigma_y = -5$ MPa

$\tau_{xy} = 2$ MPa

The stresses along the principal material directions can be determined using stress transformation formula given by,

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (3.265)$$

For $\theta = 30^o$

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} 0.75 & 0.25 & 0.866 \\ 0.25 & 0.75 & -0.866 \\ -0.433 & 0.433 & 0.5 \end{bmatrix} \begin{Bmatrix} 5 \\ -5 \\ 2 \end{Bmatrix}$$

$\sigma_L = 4.23$ MPa

$\sigma_T = -4.23$ MPa

$\tau_{LT} = -3.33$ MPa

The strains along the principal material directions can be determined from the strain-stress relations:

$$\varepsilon_L = \frac{\sigma_L}{E_L} - \nu_{TL}\frac{\sigma_T}{E_T} \qquad (3.266)$$

$$= \frac{4.23}{13.8\times10^3} - 0.087*\frac{-4.23}{3.35\times10^3}$$

$$= 4.164 \text{ x } 10^{-4}$$

$$\varepsilon_T = \frac{\sigma_T}{E_T} - \nu_{LT}\frac{\sigma_L}{E_L} \qquad (3.267)$$

$$\varepsilon_T = \frac{-4.23}{3.35\times10^3} - 0.36*\frac{4.23}{13.8\times10^3}$$

$$= -13.73 \text{ x } 10^{-4}$$

$$\gamma_{LT} = \frac{\tau_{LT}}{G_{LT}} \qquad (3.268)$$

$$\gamma_{LT} = \frac{-3.33}{4.12\times10^3}$$

$$= -8.083 \text{ x } 10^{-4}$$

Now, the strains along the xy directions can be calculated from the strain transformation law.

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & -\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & \sin\theta\cos\theta\\ 2\sin\theta\cos\theta & -2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}\begin{Bmatrix}\varepsilon_L\\ \varepsilon_T\\ \gamma_{LT}\end{Bmatrix} \qquad (3.269)$$

For $\theta = 30^o$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}0.75 & 0.25 & -0.433\\ 0.25 & 0.75 & 0.433\\ 0.866 & -0.866 & 0.5\end{bmatrix}\begin{Bmatrix}4.164\times10^{-4}\\ -13.73\times10^{-4}\\ -8.083\times10^{-4}\end{Bmatrix}$$

$$\varepsilon_x = 3.19 \text{ x } 10^{-4}$$

$$\varepsilon_y = -12.76 \text{ x } 10^{-4}$$

$$\gamma_{xy} = 11.45 \text{ x } 10^{-4}$$

Problem 3.6: A shear stress τ_{xy} = -15 MPa is applied on a unidirectional angle-ply lamina. The fibers are at 45° to the x-axis. Calculate the stresses in the principal material directions.

Solution:

As $\sigma_x = \sigma_y = 0$, the stress transformation equations become,

$$\sigma_L = \sigma_x \cos^2\theta + \sigma_y \sin 2\theta + 2\tau_{xy} \sin\theta \cos\theta \qquad (3.270)$$

$$\therefore \sigma_L = 2\tau_{xy} \sin\theta \cos\theta \qquad (3.271)$$

$$= 2*(-15)\sin 45^\circ \cos 45^\circ$$

$$= -15\ MPa$$

$$\sigma_T = \sigma_x \sin^2\theta + \sigma_y \cos 2\theta - 2\tau_{xy} \sin\theta \cos\theta \qquad (3.272)$$

$$\therefore \sigma_T = -2\tau_{xy} \sin\theta \cos\theta \qquad (3.273)$$

$$= -2*(-15)\sin 45^\circ \cos 45^\circ$$

$$= 15\ M$$

$$\tau_{LT} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}\left(\cos 2\theta - \sin 2\theta\right) \qquad (3.274)$$

$$\therefore \tau_{LT} = \tau_{xy}\left(\cos 2\theta - \sin 2\theta\right) \qquad (3.275)$$

$$= (-15)\left(\cos 90^\circ - \sin 90^\circ\right)$$

$$= 15$$

From the values found, it is clear that the principal material directions will be the principal stress axes under these loading condition.